GENERALIZED WEIERSTRASS RELATIONS AND FROBENIUS RECIPROCITY

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ABSTRACT. This article investigates local properties of the further generalized Weierstrass relations for a spin manifold S immersed in a higher dimensional spin manifold M from viewpoint of study of submanifold quantum mechanics. We show that kernel of a certain Dirac operator defined over S, which we call submanifold Dirac operator, gives the data of the immersion. In the derivation, the simple Frobenius reciprocity of Clifford algebras S and M plays important roles.

1. INTRODUCTION

This article is a sequel of the previous paper [22]. We study a connection between the generalized Weierstrass relation and Frobenius reciprocity, which is partially described in [22], and obtain further generalized Weierstrass relation over a spin manifold S immersed in higher dimensional spin manifold M.

The generalized Weierstrass relation is a generalization of the Weierstrass relation appearing in the minimal surface theory [7], which gives data of immersion of a conformal surface in higher dimensional flat spaces, e.g., euclidean space. Although similar relations appeared in [7] and it was obtained by K. Kenmotsu [16], the generalized Weierstrass relation was mainly studied in 1990's, by B. G. Konopelchenko [17] and G. Landolfi [18], F. Pedit and U. Pinkall [28], I. A. Taimanov [30], and so on. Their studies are, basically, in the framework of geometrical interpretations of integrable system. In the studies a certain Dirac operator appears and its global solutions of its Dirac equation provides the data of immersion of surfaces; in this article, we shall call, later, the Dirac operator (equation) *submanifold Dirac operator (equation)*. T. Friedrich investigated the relations for a surface immersed in the euclidean 3-space $\mathbb{R}^3$ from a viewpoint of study of Dirac operator [9]. V. V. Varlamov also studied the relations from a point of view of Clifford algebra [33]. In [18], surfaces in flat n-space are treated and those in Riemann spaces were mentioned. Further L. V. Bogdanov and E. V. Ferapontov generalized the relation to a surface in projective space [3]. Recently I. A. Taimanov gave a proper survey on the related topics and open problems in [32].

The author has studied the submanifold Dirac operator since 1990 in the framework of quantum mechanics over submanifolds, which we call submanifold quantum mechanics; in the framework, we deal with a restriction of differential operator, hamiltonian, defined over a manifold to one over its submanifold and then we find a non-trivial structure in the operator due to the half-density [13, 23]. In [26, 19] he and his coauthor investigated the Dirac operator over curves in flat space and showed that the Dirac operator is identified with the operator of the Frenet-Serret relation and a natural linear operator in the soliton theory. The latter one gives a geometrical interpretation of integrable system. When we apply the scheme developed in [26, 19] to the immersed surface case [[20] and reference therein], we also encounter the same situation; the Dirac operator coincides with the Dirac operator appearing in the generalized Weierstrass relations and with a natural linear operator of an two-dimensional soliton equation. Further the analytic torsion of the submanifold Dirac operator is also connected with globally geometrical properties [20, 21], as the Dirac operator with gauge fields exhibits geometrical properties of its related principal bundle via the analytic torsion in the framework of the Atiyah-Singer index theorem and so on [4]; the submanifold Dirac operator is also directly associated with the global geometry.

In the series of works, the author has considered why the Dirac operator given in the framework of the submanifold quantum mechanics appears in the generalized Weierstrass relation and expresses geometrical properties of submanifold. In other words, our motivation of the study is to clear what is the submanifold quantum mechanics and what is the generalized Weierstrass relation from viewpoint of study of the submanifold quantum mechanics.

In fact, recently shape effect in quantum mechanics becomes to play a more important role in physics due to development of nanotechnology. The submanifold Schrödinger operator in the submanifold quantum mechanics is applied to more actual geometrical objects [8, 24, 12, 25]. Thus it is required to reveal mathematical (analytic, geometrical and algebraic) structure of the submanifold quantum mechanics. On the submanifold Schrödinger operator, its algebraic essential was clarified [23].

This article is the final version of the studies on the construction and the local properties of the submanifold Dirac operators. We find the answers to the problem why the submanifold Dirac operator constructed in the framework of the submanifold quantum mechanics represents immersed geometry; this means a local aspect of the generalized Weierstrass relation. Though, of course, the global feature of the Dirac equation might be more interesting than local ones, its essential of the answer is based on local properties, which are connected with the simple Frobenius relations in a local chart. Thus it is not difficult to generalize the submanifold Dirac operator defined over a surface immersed in $\mathbb{R}^n$ to one over more general geometrical situations, at least locally. If there is no obstruction, it might determine a submanifold globally.

Here we note that the definition and construction of our submanifold Dirac operator differs from that of C. Bär [2], though both forms coincide. On the line, N. Ginoux and B. Morel [10], and H. d'Oussama and X. Zhang [27] also investigated eigenvalues of the submanifold Dirac operator. However their construction is not directly associated with our requirement. Thus we concentrate ourselves into the reveal using the scheme of the submanifold quantum mechanics.

We will mention our plan of this article. Section 2 shows our conventions of the Clifford algebra whereas section 3 provides our geometrical assumptions and conventions of this article. After we consider the Dirac operator over a manifold in section 4, we will construct a Dirac operator over its submanifold and investigate it in section 5. There we will give our main theorem as Theorem 5.1.

2. Local expression of Clifford Algebra

In this section, in order to show our convention in this article, we will briefly review the Clifford algebra [1, 5, 11]. The Clifford Algebra $\mathrm{CLIFF}(\mathbb{R}^m)$ is introduced as a quotient ring of a tensor algebra, $\mathbb{T}(\mathbb{R}^m)/((v, u)_{\mathbb{R}^m} - 1)$, where u, v are elements of m-dimensional vector space $\mathbb{R}^m$ and $(v, u)_{\mathbb{R}^m}$ is the natural inner product.

With respect to the degree of a tensor product, we have a natural filtration $\mathcal{F}^\ell \mathrm{CLIFF}(\mathbb{R}^m) \supset \mathcal{F}^{\ell-1}\mathrm{CLIFF}(\mathbb{R}^m)$, where $\mathcal{F}^0\mathrm{CLIFF}(\mathbb{R}^m) = \mathbb{R}$ and $\mathcal{F}^p\mathrm{CLIFF}(\mathbb{R}^m) = 0$ for $p < 0$, with a graded algebra $\mathrm{CLIFF}^p(\mathbb{R}^m) := \mathcal{F}^p\mathrm{CLIFF}^1(\mathbb{R}^m)/\mathcal{F}^{p-1}\mathrm{CLIFF}^1(\mathbb{R}^m)$. Let its subalgebra with even degrees be denoted by $\mathrm{CLIFF}^{\mathrm{even}}(\mathbb{R}^m) = \cup_{p=\mathrm{even}}^m \mathrm{CLIFF}^p(\mathbb{R}^m)$.

The exterior algebra $\wedge \mathbb{R}^m = \oplus_{j=1}^m \wedge^j \mathbb{R}^m$, is isomorphic to $\mathrm{CLIFF}(\mathbb{R}^m)$ as $\mathbb{R}^m$ vector space, $\wedge^p \mathbb{R}^m \to \mathrm{CLIFF}^p(\mathbb{R}^m)$ and thus let the isomorphism,

$$(2.1) \qquad \gamma^{(m)} : \mathbb{R}^m \to \mathrm{CLIFF}^1(\mathbb{R}^m).$$

For the basis of $\mathbb{R}^m$ denoted by $(e^{(m),i})_{i=1,\cdots,m}$, let $*$ operator be the involution in $\mathrm{CLIFF}(\mathbb{R}^m)$ such that $(\gamma^{(m)}(e^{(m),i_1}) \cdots \gamma^{(m)}(e^{(m),i_j}))^* := (\gamma^{(m)}(e^{(m),i_j}) \cdots \gamma^{(m)}(e^{(m),i_1}))$.

Let $\mathrm{Cliff}(\mathbb{R}^m)$ be a left $\mathrm{CLIFF}(\mathbb{R}^m)$-module whose endomorphism $\mathrm{END}(\mathrm{Cliff}(\mathbb{R}^m))$ is isomorphic to $\mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^m)$ ($\equiv \mathrm{CLIFF}(\mathbb{R}^m) \otimes \mathbb{C}$) as $2^{[n/2]}$ dimensional $\mathbb{C}$-vector space representation; $\epsilon_m : \mathrm{CLIFF}^{\mathbb{C}}(\mathbb{R}^m) \to \mathrm{END}(\mathrm{Cliff}(\mathbb{R}^m))$. Let $\mathrm{Cliff}^*(\mathbb{R}^m)$ be a right $\mathrm{CLIFF}(\mathbb{R}^m)$-module which is isomorphic to $\mathrm{Cliff}(\mathbb{R}^m)$; $\varphi : \mathrm{Cliff}(\mathbb{R}^m) \to \mathrm{Cliff}(\mathbb{R}^m)^*$; for $C \in \mathrm{CLIFF}(\mathbb{R}^m)$ and $c \in \mathrm{Cliff}(\mathbb{R}^m)$, $\varphi(Cc) = \varphi(c)C^*$ and let $\overline{c} := \varphi(c)$.

We may find bases $(c^{(m),a})_{a=1,\cdots,2^{[m/2]}} \in \mathrm{Cliff}(\mathbb{R}^m)$ such that for $\overline{c^{(m),a}} = \varphi(c^{(m),a})$, $\overline{c^{(m),a}}c^{(m),b} = \delta_{a,b}$. Every $\psi^{(m)} \in \mathrm{Cliff}(\mathbb{R}^m)$ is expressed as $\psi^{(m)} = \sum_{a=1}^{2^{[m/2]}} \psi_a^{(m)} c^{(m),a}$. For $\overline{\phi^{(m)}} = \sum_{a=1}^{2^{[m/2]}} \overline{\phi_a^{(m)}} \, \overline{c^{(m),a}}$ and $\psi^{(m)}$, we will

introduce a natural pairing:

(2.2) $$\langle\rangle_{\mathrm{Cliff}(\mathbb{R}^m)} : \mathrm{Cliff}(\mathbb{R}^m)^* \times \mathrm{Cliff}(\mathbb{R}^m) \to \mathbb{C},$$

by

$$\langle \overline{\phi^{(m)}}, \psi^{(m)} \rangle_{\mathrm{Cliff}(\mathbb{R}^m)} = \sum_{a=1}^{2^{[m/2]}} \overline{\phi_a^{(m)}} \psi_a^{(m)}.$$

For multiplicative group of $\mathrm{CLIFF}^{\mathrm{even}}(\mathbb{R}^m)$, $\mathrm{CLIFF}^{\mathrm{even},\times}(\mathbb{R}^m)$, the Clifford group $\mathrm{CG}(\mathbb{R}^m)$ is defined by

$$\{\tau \in \mathrm{CLIFF}^{even\times}(\mathbb{R}^m) \mid \text{ for } \forall v \in \mathrm{CLIFF}^1(\mathbb{R}^m), \tau v \tau^* \in \mathrm{CLIFF}^1(\mathbb{R}^m)\}.$$

For representations ϵ_m and ϵ_m', there exists $\tau \in \mathrm{CG}(\mathbb{R}^m)$ and an action A_τ on ϵ's such that $A_\tau \epsilon_m(C) = \epsilon_m'(\tau C \tau^{-1})$ for $C \in \mathrm{CLIFF}(\mathbb{R}^m)$. Due to $\gamma^{(m)} : \mathbb{R}^m \to \mathrm{CLIFF}^1(\mathbb{R}^m)$ and (2.2), we have

(2.3) $$\langle \, , \, \rangle_{\mathrm{Cliff}(\mathbb{R}^m)} : \mathrm{Cliff}(\mathbb{R}^m)^* \times \gamma^{(m)}(\mathbb{R}^m) \times \mathrm{Cliff}(\mathbb{R}^m) \to \mathbb{C}.$$

This is a linear map from $\mathbb{R}^m$ to $\mathbb{C}$. Let the coproduct be $\mathfrak{m} : \mathrm{Cliff}(\mathbb{R}^m) \to \mathrm{Cliff}(\mathbb{R}^m) \times \mathrm{Cliff}(\mathbb{R}^m)$, $(\psi \mapsto (\psi, \psi))$. Restricted domain to its inverse image of $\mathbb{R} \subset \mathbb{C}$, (2.3) with the operator "$\langle, \gamma^{(m)}(\cdot) \, \rangle \circ \varphi \otimes 1 \circ \mathfrak{m}$" can be regarded as $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^m, \mathbb{R}) \approx \mathbb{R}^m$. Hence we have the following lemma.

Lemma 2.1. *There exists a subset $\mathrm{Cliff}^{pr}(\mathbb{R}^m)$ of $\mathrm{Cliff}(\mathbb{R}^m)$ which is isomorphic to $\mathbb{R}^m$ as $\mathbb{R}$-vector space such that for*

$$\mathfrak{i} : \mathbb{R}^m \to \mathrm{Cliff}^{pr}(\mathbb{R}^m) \subset \mathrm{Cliff}(\mathbb{R}^m), \quad (v \mapsto \psi_v^{pr}),$$

$$\mathfrak{j} := \langle \, , \gamma^{(m)}(\cdot) \, \rangle \circ \varphi \otimes 1 \circ \mathfrak{m} \otimes 1 \circ \mathfrak{i} \otimes 1 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

is identified with the inner product $(,)_{\mathbb{R}^m} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$, $(u,v)_{\mathbb{R}^m} \equiv \mathfrak{j}(u,v)$, i.e.,

$$\langle \overline{\psi_v^{pr}}, \gamma^{(m)}(w) \psi_v^{pr} \rangle_{\mathrm{Cliff}(\mathbb{R}^m)} = (v, w)_{\mathbb{R}^m}.$$

This lemma shows that there exist elements $(\phi_{e^{(m),j}}^{(m)})_{j=1,\cdots,m}$ of $\mathrm{Cliff}^{pr}(\mathbb{R}^m)$; for $b^{(m),i} = \sum_{j=1}^m \Lambda^i{}_j e^{(m),j}$,

(2.4) $$\langle \overline{\phi}_{e^{(m),\ell}}^{(m)}, \gamma^{(m)}(b^{(m),i}) \phi_{e^{(m),\ell}}^{(m)} \rangle_{\mathrm{Cliff}(\mathbb{R}^m)} = \Lambda^i{}_\ell.$$

This correspondence is well-known in physicists, which is, of course, independent from the coordinate system and gives the data of $\mathrm{SO}(\mathbb{R}^m) \times \mathbb{R}$.

Here let us consider an embedding $\mathbb{R}^k$ into $\mathbb{R}^n$ ($k < n$): $\iota_{n,k} : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$ and $\pi_{k,n} : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ such that for $u^{(n)} \in \mathbb{R}^n$ and $v^{(k)} \in \mathbb{R}^k$, $(\iota_{n,k} u^{(n)}, v^{(k)})_{\mathbb{R}^n} \equiv (u^{(n)}, \pi_{n,k} v^{(k)})_{\mathbb{R}^k}$. In this article, we are concerned with the moduli of the embedding or Grassmann manifold $\mathrm{Gr}_{n,k} := \mathrm{SO}(n)/\mathrm{SO}(k)\mathrm{SO}(n-k)$. The embedding $\iota_{n,k}$ corresponds to a point q of $\mathrm{Gr}_{n,k} = \mathrm{SO}(\mathbb{R}^n)/\mathrm{SO}(\mathbb{R}^n)\mathrm{SO}(\mathbb{R}^{n-k})$. Using the Clifford module we will deal with them like [1]. The following proposition is obvious due to [1].

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Proposition 2.1. *(1) For $k < n$, $\mathrm{CLIFF}(\mathbb{R}^k)$ is a subalgebra $\mathrm{CLIFF}(\mathbb{R}^n)$ by the natural inclusion of generators. $\iota_{n,k}^{\flat} : \mathrm{CLIFF}(\mathbb{R}^k) \to \mathrm{CLIFF}(\mathbb{R}^n)$.*
(2) For $k < n$, $\mathrm{CG}(\mathbb{R}^k)$ is a natural subgroup of $\mathrm{CG}(\mathbb{R}^n)$.

The $\iota_{n,k}$ and $\pi_{k,n}$ give an induced representation and a restriction representation: There exists an element τ_q in $\mathrm{CG}(\mathbb{R}^n)$ such that

$$
\begin{aligned}
\mathrm{Ind}^{\tau_q}{}^n_k &\; : \mathrm{Cliff}(\mathbb{R}^k) \to \mathrm{Cliff}(\mathbb{R}^n), &&\left(\mathrm{Ind}^{\tau_q}{}^n_k \psi^{(k)} := \sum_{a=1}^{2^{[k/2]}} \psi^{(k)}_a \left[\sum_{b=1}^{2^{[n/2]}} \tau_q{}^a_{\ b} c^{(n),b} \right] \right), \\
\mathrm{Res}^{\tau_q}{}^n_k &\; : \mathrm{Cliff}(\mathbb{R}^n) \to \mathrm{Cliff}(\mathbb{R}^k), &&\left(\mathrm{Res}^{\tau_q}{}^n_k \psi^{(n)} := \sum_{a=1}^{2^{[k/2]}} \left[\sum_{b=1}^{2^{[n/2]}} \psi^{(n)}_b \tau_q^{-1}{}^b_{\ a} \right] c^{(k),a} \right).
\end{aligned}
$$

The Frobenius reciprocity gives for $\psi^{(k)} \in \mathrm{Cliff}(\mathbb{R}^k)$ and $\phi^{(n)} \in \mathrm{Cliff}(\mathbb{R}^n)$,

$$(2.5) \qquad \langle \overline{\mathrm{Res}^{\tau_q}{}^n_k \psi^{(n)}}, \phi^{(k)} \rangle_{\mathrm{Cliff}(\mathbb{R}^k)} = \langle \psi^{(n)}, \overline{\mathrm{Ind}^{\tau_q}{}^n_k \phi^{(k)}} \rangle_{\mathrm{Cliff}(\mathbb{R}^n)}.$$

Using the relation (2.5), we will consider Λ in (2.4) and its relation to the point q of the Grassmannian $\mathrm{Gr}_{n,k}$. For $u^{(n)} \in \mathbb{R}^n$ and $v^{(k)} \in \mathbb{R}^k$, let $\psi^{(n)}_{u^{(n)}} = \mathrm{i}(u^{(n)})$ and $\psi^{(k)}_{u^{(n)}} := \mathrm{Res}^{\tau_q}{}^n_k \psi^{(n)}_{u^{(n)}}$ using $\tau_q \in \mathrm{CG}(\mathbb{R}^n)$, and then we have the relation, $\gamma^{(n)}(\iota_{n,k}(v^{(k)})) \psi^{(n)}_{u^{(n)}} = \mathrm{Ind}^{\tau_q}{}^n_k \gamma^{(k)}(v^{(k)}) \psi^{(k)}_{u^{(n)}}$. The Frobenius reciprocity (2.5) gives

$$(2.6)$$
$$
\begin{aligned}
\langle \overline{\psi^{(k)}_{u^{(n)}}}, \; \gamma^{(k)}(v^{(k)}) \psi^{(k)}_{u^{(n)}} \rangle_{\mathrm{Cliff}(\mathbb{R}^k)} &= \langle \overline{\psi^{(n)}_{u^{(n)}}}, \; \mathrm{Ind}^{\tau_q}{}^n_k \gamma^{(k)}(v^{(k)}) \psi^{(k)}_{u^{(n)}} \rangle_{\mathrm{Cliff}(\mathbb{R}^n)} \\
&= \langle \overline{\psi^{(n)}_{u^{(n)}}}, \; \gamma^{(n)}(\iota_{n,k}(v^{(k)})) \psi^{(n)}_{u^{(n)}} \rangle_{\mathrm{Cliff}(\mathbb{R}^n)} \\
&= (\iota_{n,k}(v^{(k)}), u^{(n)})_{\mathbb{R}^n}.
\end{aligned}
$$

Every pair $(u^{(n)}, v^{(k)})$ recovers the point q in $\mathrm{Gr}_{n,k}$.

This relation (2.6) has an alternative expression using another reference embedding $\iota_{n,k}^o : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$ associated to a base point of $o \in \mathrm{Gr}_{n,k}$ and $\tau_o \in \mathrm{CG}(\mathbb{R}^n)$. For given τ_q and τ_o of $\mathrm{CG}(\mathbb{R}^n)$, we find an element $\tau \in \mathrm{CG}(\mathbb{R}^n)$ such that $\tau_q = \tau^{-1} \tau_o$. When one wishes to consider τ_q as a representation of $\mathrm{Gr}_{n,k}$, he could deal with the element τ by fixing τ_o; and investigate $\mathrm{Gr}_{n,k}$. Then we have

$$\mathrm{Ind}^{\tau_q}{}^n_k = \tau^{-1} \mathrm{Ind}^{\tau_o}{}^n_k, \quad \mathrm{Res}^{\tau_q}{}^n_k = \mathrm{Res}^{\tau_o}{}^n_k \tau.$$

For the situations of (2.6), let $\phi^{(n)}_{u^{(n)}} := \tau \psi^{(n)}_{u^{(n)}}$ and then we have $\psi^{(n)}_{u^{(n)}} = \mathrm{Res}^{\tau_o}{}^n_k \phi^{(n)}_{u^{(n)}}$. (2.6) becomes

$$(2.7) \quad
\begin{aligned}
\langle \overline{\psi^{(k)}_{u^{(n)}}}, \; \gamma^{(k)}(v^{(k)}) \psi^{(k)}_{u^{(n)}} \rangle_{\mathrm{Cliff}(\mathbb{R}^k)} &= \langle \overline{\phi^{(n)}_{u^{(n)}}}, \; \gamma^{(n)}(\iota_{n,k}^o(v^{(k)})) \phi^{(n)}_{u^{(n)}} \rangle_{\mathrm{Cliff}(\mathbb{R}^n)} \\
&= (\iota_{n,k}(v^{(k)}), u^{(n)})_{\mathbb{R}^n}.
\end{aligned}
$$

This also provides the data of $\mathrm{Gr}_{n,k}$ and the immersion, which essentially comes from (2.5) and Lemma 2.1. We will use latter relation (2.7) for the generalized Weierstrass relations.

3. Geometrical Preliminary

In this section, we will give a geometrical preliminary. As we use primitive facts in sheaf theory [15], first we show our conventions as follows. For a fiber bundle A over a paracompact differential manifold X and an open set $U \subset X$, let A_X denote a sheaf given by a set of smooth local sections of the fiber bundle A, e.g., $\mathbb{C}_X^r$ is a sheaf given by smooth local sections of complex vector bundle over X of rank r, and $A_X(U) \equiv \Gamma(U, A_X)$ sections of A_X over U.

Further for open sets $U \subset V \subset X$, the restriction of a sheaf A_X is denoted by r_{UV}. Using the direct limit for $\{U \mid pt \in U \subset X\}$, we have a *stalk* A_{pt} of A_X by setting $A_{pt} \equiv \Gamma(pt, A_X) := \lim_{U \to pt} A_X(U)$. Similarly for a compact subset K in X, $i_K : K \hookrightarrow X$ and for $\{U \mid K \subset U \subset X\}$, we have $\Gamma(K, A_X) := \lim_{U \to K} A_X(U)$ and $r_{K,U} A_X$.

On the other hand, for a topological subset Y of X, $i_Y : Y \hookrightarrow X$, there is an inverse sheaf, $i_Y^{-1} A_X$ given by the sections $i_Y^{-1} A(U) = \Gamma(i_Y(U), A_X)$ for $U \subset Y$. When Y is a compact set, we have an equality $\Gamma(i_Y Y, A_X) = \Gamma(Y, i_Y^{-1} A_X)$ (Theorem 2.2 in [15]) and we identify them in this article. Further $\Gamma_c(U, A_X)$ denotes the set of smooth sections of A_X whose support is compact in U. For a compact subset K of X, $\Gamma_K(X, A_X)$ is a set of global sections of A_X whose support is in K.

Let (M, g_M) be a n spin manifold, which is acted by a Lie transformation group G as its isometory. The metric g_M of M is a global section of sheaf $\mathrm{Hom}_{\mathbb{R}}(\Theta_M, \Omega_M)$, where Θ_M and Ω_M are sheaves of tangent and cotangent spaces as $\mathcal{C}^\infty$-modules: $g_M(\ ,\ ) : \Theta_M \times \Theta_M \to \mathbb{R}_M$.

Let us consider a locally closed k spin manifold S embedded in M [34, 14]; $\iota_{M,S} : S \hookrightarrow M$, so that for every point p in S, there is a subgroup H of G satisfying

$$(3.1) \qquad\qquad T_p M = T_p(H \circ p) \oplus T_p S.$$

We identify $\iota_{M,S}(S)$ with S. H may depend on the position p in general.

Since $\iota_{M,S}{}^{-1}\Theta_M$ can be regarded as a subsheaf of the (n, k) Grassmannian sheaf $\mathrm{Gr}_S^{(n,k)}$ over S, fixing a section $\mathrm{Gr}_S^{(n,k)}$ corresponds to determine the immersion $\iota_{M,S}$ up to global symmetry like euclidean moves. We consider $\iota_{M,S}{}^{-1}\Theta_M$ and $\iota_{M,S_*}\Theta_S$. Let $\Theta_S^\perp := \iota_{M,S}{}^{-1}\Theta_M/\Theta_S$; $\mathrm{Gr}_S^{(n,k)}$ can be realized as the quotient of orthogonal group sheaves $\mathrm{Gr}_S^{(n,k)} = \mathrm{SO}(\iota_{M,S}{}^{-1}\Theta_M)/\mathrm{SO}(\Theta_S)\mathrm{SO}(\Theta_S^\perp)$.

For example, as $r_{S,M}$ is defined by a direct limit of open sets of M to $\iota_{M,S}(S)$. we should consider its vicinity in M. We prepare a tubular neighborhood T_S of S in M; $\pi_{S,T_S} : T_S \to S$ and $\iota_{M,T_S} : T_S \hookrightarrow M$.

As our theory is local and we use only germs at a point in vicinity of $\iota_{M,S}(S)$, we consider a sufficiently small open set U in M such that $U \cap S \neq \emptyset$ instead of M and S; Without loss of generality, we assume that M and S are diffeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, there exists a compact subset K of M such that $S \subset K$, and later we may sometimes identify M with T_S.

Further due to the group action H, we assume that T_S and S satisfy the following conditions.

(1) T_S behaves as a normal bundle $\pi_{S,T_S} : T_S \to S$,
(2) there exist the base $b^{(n),\dot\alpha}$ ($\dot\alpha = k+1, \cdots, n$) of T_S and $\Theta_S^\perp$, its dual base $b_{\dot\alpha}^{(n)}$, and $q := (q^{\dot\alpha})_{\dot\alpha = k+1, \cdots, n}$ the normal coordinate of T_S such that 1) for $X \in \Theta_S(S)$ and the Riemannian connection ∇_X in M, $\nabla_X b_{\dot\alpha}^{(n)}$ belongs to $\Theta_S(S)$ (See proof of Lemma 3.1) and 2) every point $pt \in T_S$ is expressed by $pt = \pi_{S,T_S}pt + q^{\dot\alpha} b_{\dot\alpha}^{(n)}$.
(3) T_S and S have local parameterization. $u : T_S \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that $u = (s,q)$ and $s : S \to \mathbb{R}^k$; $u = (u^\mu)_{\mu=1,\cdots,n} = (s^\alpha, q^{\dot\alpha})_{\alpha=1,\cdots,k,\dot\alpha=k+1,\cdots,n}$. (We will use the Einstein convention.)

Let S_q be $u^{-1}(\mathbb{R}^k \times \{q\})$ for fixing q. $\{S_q\}_q$ has a foliation structure. As a result of (3.1), S could be interpreted as an analytic manifold;

$$S \equiv \{(s,q) \in T_S \mid q = 0\}.$$

For every sheaf A_{T_S} of T_S, we have a sheaf A_S of S and a restriction map $r_{S,M} : A_{T_S}^n \to A_S^n$ by substituting $q = 0$ into $f(s,q)$. Hereafter we use the symbol $r_{S,M}$ in this meaning. Due to the above assumption, the metric g_{T_S} of T_S at (s,q) induced from M is given as

$$(3.2) \qquad g_{T_S} = \begin{pmatrix} g_{S_q} & 0 \\ 0 & 1 \end{pmatrix},$$

where g_{S_q} is a metric S_q given by proof of Lemma 3.1. We also introduce objects and maps for S_q as for S, e.g., ι_{M,S_q}.

Lemma 3.1. *Let g_{T_S} and g_S be induced metrics of g_M and $\Gamma_{\dot\alpha}/k$ be the mean curvature vector field along $b^{(n)\dot\alpha}$ [[34] p.119],*

$$(3.3) \qquad \det g_{T_S} = \rho \det g_S, \quad \rho = (1 + \Gamma_{\dot\alpha}q^{\dot\alpha} + \mathcal{O}(q^{\dot\alpha}q^{\dot\beta}))^2.$$

Proof. In general, we consider more general normal unit vectors $\tilde{b}_{\dot\alpha}^{(n)} \in T_{pt}^\perp S$ at $pt \in S$. At a point in S, we find the Christoffel symbol $\Gamma^\beta_{\dot\beta\alpha}$ over S as a relation, the equation of Weingarten [[34] p.119],

$$(3.4) \qquad \nabla_\alpha \tilde{b}_{\dot\beta}^{(n)} = \Gamma^\beta_{\dot\beta\alpha} b_\beta^{(n)} + \tilde{\Gamma}^{\dot\alpha}_{\alpha\dot\beta} \tilde{b}_{\dot\alpha}^{(n)}.$$

Here ∇_α is the Riemannian connection of M for the direction $\partial/\partial s^\alpha$ of TS, and $b_\beta^{(n)} := \partial/\partial s^\beta$ of TS. Let $\Lambda_{\dot\alpha}^{\dot\beta}$ be a section of $SO(\Theta_S^\perp)$ such that its Lie algebraic parameter $\theta_{\dot\alpha,\dot\beta}$ satisfies $\partial_\alpha \theta_{\dot\alpha,\dot\beta} = \tilde{\Gamma}^{\dot\alpha}_{\alpha\dot\beta}$ noting $\tilde{\Gamma}^{\dot\alpha}_{\alpha\dot\beta} = -\tilde{\Gamma}^{\dot\beta}_{\alpha\dot\alpha}$. Since

S is homeomorphic to $\mathbb{R}^k$, we can find such a parameter $\theta_{\dot\alpha,\dot\beta}$ by solving the differential equation.

Let $b^{(n)}_{\dot\alpha} = \Lambda^{\dot\beta}_{\dot\alpha}\tilde{b}^{(n)}_{\dot\beta}$. Then (3.4) is reduced to

$$\nabla_\alpha b^{(n)}_{\dot\beta} = \Gamma^\beta_{\dot\beta\alpha} b^{(n)}_\beta.$$

For a point pt in T_S, the moving frame $e^{(n),i} = dx^i \in \Gamma(pt, \Theta_{T_S})$ is expressed by $e^{(n),i}_\alpha ds^\alpha = (\pi_{S,T_S*}(e^i_\alpha) + q^{\dot\alpha}\Gamma^\beta_{\dot\alpha\alpha}b^{(n),i}_\beta)ds^\alpha$. The metric in T_S and its determinant are given by

$$(3.5)\qquad g_{S_q\alpha\beta} = g_{S\alpha\beta} + [\Gamma^\Gamma_{\dot\alpha\alpha}g_{S\gamma\beta} + g_{S\alpha\gamma}\Gamma^\gamma_{\dot\alpha\beta}]q^{\dot\alpha} + [\Gamma^\delta_{\dot\alpha\alpha}g_{S\delta\gamma}\Gamma^\gamma_{\dot\beta\beta}]q^{\dot\alpha}q^{\dot\beta},$$

where $g_{S\alpha\beta} := g_{M\,i,j}e^i_\alpha e^j_\beta$; Let $\Gamma_{\dot\beta} := \Gamma^\alpha_{\dot\beta\alpha}$ over S; $(\Gamma_{\dot\beta})/k$ is the mean curvature vector of $b^{(n)\dot\beta}$ [[34] p.119]. $\qquad\square$

There is an action of $\mathrm{SO}(\Theta^\perp_S)$ on $\Theta^\perp_S$. Obviously (3.3) is invariant for the action $\mathrm{SO}(\Theta^\perp_S)$.

4. Dirac System in M

For the above geometrical situation, we will consider a Dirac equation over M [[4] 3.3] here.

We, first, introduce a paring given by the pointwise product $\langle,\rangle_{\mathrm{Cliff}_M}$ for the germs of the Clifford module Cliff_M over M and its natural hermite conjugate Cliff^*_M; φ_{pt} is the hermite conjugate operator which gives the isomorphism from Cliff_M to Cliff^*_M and $\langle\overline{\psi}_{M,1}\psi_{M,2}\rangle_{\mathrm{Cliff}_M} \in \Gamma(pt, \mathbb{C}_M)$.

We deal with a Dirac equation over M as an equation over another pre-Hilbert space $\mathcal{H} = (\Gamma_c(M, \mathrm{Cliff}^*_M) \times \Gamma_c(M, \mathrm{Cliff}_M), \langle,\rangle, \varphi)$. Here $\langle,\rangle_M$ is the L^2-type pairing, for $(\overline{\psi}_{M,1}, \psi_{M,2}) \in \Gamma_c(M, \mathrm{Cliff}^*_M) \times \Gamma_c(M, \mathrm{Cliff}_M)$,

$$(4.1)\qquad \langle\overline{\psi}_{M,1}, \psi_{M,2}\rangle_M = \int_M d\mathrm{vol}_M\ \langle\overline{\psi}_{M,1}, \psi_{M,2}\rangle_{\mathrm{Cliff}_M}$$

Here in T_S, the measure of M is decomposed to

$$(4.2)\qquad d\mathrm{vol}_M = \rho(\det g_S)^{1/2}d^k s d^{n-k}q,$$

$d^k s = \wedge^s_{\alpha=1}ds_\alpha$, and $d^{n-k}q = \wedge^n_{\dot\alpha=k+1}dq_{\dot\alpha}$. Further in this article, we express the preHilbert space using the triplet with the inner product $(\circ, \cdot)_M := \langle\varphi\circ, \cdot\rangle_M$. For an operator P over Cliff_M, let $Ad(P)$ be defined by the relation, $\langle\overline{\psi}_1, P\psi_2\rangle_M = \langle\overline{\psi}_1 Ad(P), \psi_2\rangle_M$ if exists. Further for $\psi \in \Gamma_c(M, \mathrm{Cliff}_M)$, P^* is defined by $P^*\psi = \varphi^{-1}(\varphi(\psi)Ad(P))$.

Let the sheaf of the Clifford ring over M be denoted by CLIFF_M. As a model of (2.1) let γ_M be a morphism from Ω_M to CLIFF^1_M.

The Dirac operator is a morphism between the Clifford module

$$\slashed{D}_M : \mathrm{Cliff}_M \to \mathrm{Cliff}_M$$

but as a differential operator, we could extend its domain and region to,

$$\slashed{D}_M : \mathbb{C}_M^{2^{[n/2]}} \to \mathbb{C}_M^{2^{[n/2]}}.$$

Since $\mathrm{Cliff}_M(U)$ contains zero section, we may consider that $\mathrm{Ker}(\slashed{D}_M)$ as a subset of germs of $\mathbb{C}_M^{2^{[n/2]}}$ means a subset of germs of Cliff_M.

Then there are a set of germs $\{c_M^a\}_{a=1,\cdots,2^{[n/2]}}$ of $\mathrm{Cliff}_M(M)$ and $\overline{c_M^a} := \varphi(c_M^a)$ which hold relations at each point,

$$(4.3) \qquad \langle \overline{c}_M^a c_M^b \rangle_{\mathrm{Cliff}_M} = \delta^{a,b}, \quad \text{for } a, b = 1, \cdots, 2^{[n/2]}.$$

A germ of solutions of Dirac equation $\slashed{D}_M \psi = 0$ is expressed by $\psi = \sum_a \psi_M^a c_M^a$ for $\psi_M^a \in \Gamma(pt, \mathbb{C}_M)$ at a point $pt \in M$. Lemma 2.1 gives

Proposition 4.1. *There is a subsheaf* $\mathrm{Cliff}_M^{pr} \subset \mathrm{Cliff}_M^{pr}$ *satisfying the following:*

(1) Cliff_M^{pr} is isomorphic to Θ_M as vector sheaves via the following $\mathfrak{j}_M$, i.e., there is a morphism $\mathfrak{i} : \Theta_M \to \mathrm{Cliff}_M^{pr}$ ($\mathfrak{i}(u^{(n)}) = \psi_{u^{(n)}}$).

(2) $\mathfrak{j}_M$ whose model is $\mathfrak{j}$ in Lemma 2.1 gives an equivalence $\mathfrak{j}_M = g_M(,)$, i.e., for $v^{(n)}, u^{(n)} \in \Gamma(pt, \Theta_M)$, every $\psi_{u^{(n)}} \in \Gamma(pt, \mathrm{Cliff}_M^{pr})$ satisfies

$$\langle \overline{\psi}_{u^{(n)}} \gamma_M(g_M(v^{(n)})) \psi_{u^{(n)}} \rangle_{\mathrm{Cliff}_M} = g_M(u^{(n)}, v^{(n)}).$$

We call this relation $\mathbb{R} \times \mathrm{SO}(n)$-representation in this article.

Due to the Proposition, for $\Lambda_j^i \in \Gamma(pt, \mathrm{SO}(n) \times \mathbb{R}))$, and $v^{(n),i} := \Lambda_j^i e^{(n),j} \in \Gamma(pt, \Theta_M))$, there is a pair of germ $(\psi_{e^{(n),i}})_{i=1,\cdots,n}$ in $\Gamma(pt, \mathrm{Cliff}_M^{pr})$ of the Clifford module and its dual pair $\overline{\psi}_{e^{(n),i}} := \varphi_{pt}(\psi_{e^{(n),i}})$ which hold a relation,

$$\langle \overline{\psi}_{e^{(n),\ell}} \gamma_M(g_M(v^{(n),i})) \psi_{e^{(n),\ell}} \rangle_{\mathrm{Cliff}_M} = \Lambda_\ell^i \quad \text{(not summed over } \ell\text{)}.$$

Every sheaf A_{T_S} over T_S is determined by $A_{T_S} = r_{T_S,M} A_M$ for every sheaf A_M over M and in our conditions these properties preserves over T_S.

Remark 4.1. Using a $\mathbb{C}$-valued smooth compact function $b \in \Gamma_c(M, \mathbb{C}_M)$ over M such that $b \equiv 1$ at $U \subset M$ and its support is in M, $b\psi_M^a$, $b\psi_{e^{(n),k}}$ and their partners belong to $\Gamma_c(M, \mathrm{Cliff}_M)$ and $\Gamma_c(M, \mathrm{Cliff}_M^*)$. Hereafter we assume that ψ_M^a, $\psi_{e^{(n),k}}$ and their partners are sections of $\Gamma_c(M, \mathrm{Cliff}_M)$ and $\Gamma_c(M, \mathrm{Cliff}_M^*)$ in the sense.

The Dirac operator restricted over T_S is explicitly given by

$$(4.4) \qquad \slashed{D}_{T_S} = \gamma_{T_S}(du^\mu)(\partial_\mu + \omega_{T_S,\mu}).$$

where $\partial_\mu := \partial/\partial u^\mu$ and $\omega_{T_S,\mu}$ is a spin connection.

5. Submanifold Dirac Operator over S in M

In this section, we will define the submanifold Dirac operator over S in M and investigate its properties.

Since T_S is diffeomorphic to $\mathbb{R}^n$, $\mathbb{C}_{T_S}$ is soft (Theorem 3.1 in [15]). Hence we have the following proposition.

Proposition 5.1. Cliff_{T_S} and $\mathbb{C}_{T_S}^{2^{[n/2]}}$ are soft.

Proof. Cliff_{T_S} is considered as a sheaf of $\mathbb{C}$-vector bundle with $2^{[n/2]}$ rank. From the proof of Theorem 3.2 in [15], it is justified. $\square$

Due to the Proposition 5.1, at each point pt in S and for a germ $\psi_{pt} \in \Gamma(pt, \mathrm{Cliff}_{T_S})$, there exists $\psi_c \in \Gamma_c(T_S, \mathrm{Cliff}_{T_S})$ and $\psi_o \in \Gamma(T_S, \mathrm{Cliff}_M)$ such that $\psi_{pt} = \psi_c$ and $\psi_{pt} = \psi_o$ around pt. Thus an element of $\Gamma(pt, \mathrm{Cliff}_{T_S})$ need not be distinguished which it comes from $\Gamma_c(T_S, \mathrm{Cliff}_{T_S})$ or $\Gamma(T_S, \mathrm{Cliff}_M)$. From here, every A_{T_S} is identified with A_M again.

The action of H along the fiber direction, we will continue to consider it in the framework of the unitary representation of Clifford module and we wish to consider kernel of $\partial_{\dot\alpha} := \partial/\partial q^{\dot\alpha}$ ($\dot\alpha = k+1, \cdots, n$) [[23] and references therein]. However $p_{\dot\alpha} := \sqrt{-1}\partial_{\dot\alpha}$ is not self-adjoint, $p_{\dot\alpha}^* \neq p_{\dot\alpha}$ in general due to the existences ρ in (4.1) and (4.2).

Let us follow the techniques in the pseudo-regular representation. We introduce another preHilbert space $\mathcal{H}' \equiv (\Gamma_c(T_S, \tilde{\mathrm{Cliff}}_{T_S}^*) \times \Gamma_c(T_S, \tilde{\mathrm{Cliff}}_{T_S}), \langle,\rangle_{\mathrm{sa}}, \tilde\varphi)$ so that $p_{\dot\alpha}$'s become self-adjoint operators there. Using the half-density (Theorem 18.1.34 in [13]), we construct *self-adjointization*: $\eta_{\mathrm{sa}} : \mathcal{H} \to \mathcal{H}'$ by,

$$\eta_{\mathrm{sa}}(\overline{\psi}) := \rho^{1/4}\overline{\psi}, \quad \eta_{\mathrm{sa}}(\psi) := \rho^{1/4}\psi, \quad \eta_{\mathrm{sa}}(P) := \rho^{1/4}P\rho^{-1/4}.$$

Here since ρ does not vanish in T_S, η_{sa} gives an isomorphism $\eta_{\mathrm{sa}} : \mathrm{Cliff}_{T_S}^* \times \mathrm{Cliff}_{T_S} \to \tilde{\mathrm{Cliff}}_{T_S}^* \times \tilde{\mathrm{Cliff}}_{T_S}$. Here this transformation is also essentially the same as that in the radical Laplace operator, e.g., in Theorem 3.7 of [14][1]. For $(\overline{\psi}_1, \psi_2) \in \Gamma_c(T_S, \tilde{\mathrm{Cliff}}_{T_S}^*) \times \Gamma_c(T_S, \tilde{\mathrm{Cliff}}_{T_S})$, by letting $\tilde\varphi := \eta_{\mathrm{sa}}\varphi\eta_{\mathrm{sa}}^{-1}$, the pairing is defined by

$$(5.1) \qquad \langle \overline{\psi}_1, \psi_2 \rangle_{\mathrm{sa}} := \int_{T_S} (\det g_S)^{1/2} d^k s\, d^{n-k} q\ \langle \overline{\psi}_1, \psi_2 \rangle_{\mathrm{Cliff}_M}.$$

Here we have the properties of η_{sa} that 1) $\langle \circ, \cdot \rangle_{\mathrm{sa}} = \langle \eta_{\mathrm{sa}}^{-1}\circ, \eta_{\mathrm{sa}}^{-1}\cdot \rangle_M$, 2) for an operator P of Cliff_{T_S}, $\eta_{\mathrm{sa}}(P) = \eta_{\mathrm{sa}}P\eta_{\mathrm{sa}}^{-1}$, and 3) $p_{\dot\alpha}$'s themselves become self-adjoint in $\mathcal{H}'$, i.e., $p_{\dot\alpha} = p_{\dot\alpha}^*$.

Noting $\rho = 1$ at a point in S, for $(\overline{\psi}, \psi) \in \Gamma(S, \mathrm{Cliff}_{T_S}^*) \times \Gamma(S, \mathrm{Cliff}_{T_S})$, we have

$$r_{S,M}\eta_{\mathrm{sa}}(\overline{\psi}) = r_{S,M}\overline{\psi}, \quad \text{and} \quad r_{S,M}\eta_{\mathrm{sa}}(\psi) = r_{S,M}\psi.$$

Further we have the following proposition.

---

[1]Our $\rho^{1/2}$ corresponds to δ in p.261 in [14].

Proposition 5.2. *By letting* $p_q := a^{\dot{\alpha}} p_{\dot{\alpha}}$ *for real generic numbers* $a_{\dot{\alpha}}$*, the projection,*

$$\pi_{p_q} : \tilde{\mathrm{Cliff}}_{T_S}^{*} \times \tilde{\mathrm{Cliff}}_{T_S} \to \mathrm{Ker}(Ad(p_q)) \times \mathrm{Ker}(p_q),$$

induces the projection in the preHilbert space, i.e.,

(1) *For an open set* $U \subset T_S$*,* $\tilde{\varphi}|_{\mathrm{Ker}(p_q)} : \Gamma(U, \mathrm{Ker}(p_q)) \to \Gamma(U, \mathrm{Ker}(Ad(p_q)))$ *is isomorphic as vector space. We simply express* $\tilde{\varphi}|_{\mathrm{Ker}(p_q)}$ *by* $\tilde{\varphi}$ *hereafter.*

(2) $\mathcal{H}_{p_q} := (\Gamma_c(T_S, \mathrm{Ker}(Ad(p_q))) \times \Gamma_c(T_S, \mathrm{Ker}(p_q)), \langle, \rangle_{\mathrm{sa}}, \tilde{\varphi})$ *is a pre-Hilbert space.*

(3) $\varpi_{p_q} := \pi_{p_q}|_{\tilde{\mathrm{Cliff}}_{T_S}}$ *induces a natural restriction of pointwise multiplication for a point in* T_s*,* $\mathcal{H}_{p_q}^{pt} := (\Gamma(pt, \mathrm{Ker}(Ad(p_q))) \times \Gamma(pt, \mathrm{Ker}(p_q)), \cdot, \tilde{\varphi}_{pt})$ *becomes a preHilbert space. The hermite conjugate map* $\tilde{\varphi}_{pt}$ *is still an isomorphism.*

Proof. By letting $\varpi_{p_q} := \pi_{p_q}|_{\tilde{\mathrm{Cliff}}_{T_S}}$, we have $\varpi_{p_q} = \varpi_{p_q}^2 = \varpi_{p_q}^*$ in $\mathcal{H}_{p_q}$. In fact since $p_{\dot{\alpha}}$ is self-adjoint, $\mathrm{Ker}(p_q) = \mathrm{Ker}(p_q^*)$ and $\mathrm{Ker}(p_q)$ is isomorphic to $\mathrm{Ker}(Ad(p_q))$, i.e., $\varphi(\varpi_{p_q}\psi) = \varphi(\psi)Ad(\varpi_{p_q})$. $\varpi_{p_q}^*\psi = \varphi^{-1}(\varphi(\psi)Ad(\varpi_{p_q}))$ gives $\varpi_{p_q} = \varpi_{p_q}^*$. $\square$

Remark 5.1. We shall remark that deformation of preHilbert space by the action of η_{sa} makes ϖ_{p_q} a projection operator in the sense of $*$-algebra. This is the essential of the scheme of the submanifold quantum mechanics [23], which provides non-trivial quantum mechanics [24, 12, 25]. It is absolutely non-trivial fact but the same idea appeared in computation of Hydrogen atom in [6].

Further we consider p_q as a morphism between $\mathbb{C}_{T_S}^{2^{[n/2]}} \to \mathbb{C}_{T_S}^{2^{[n/2]}}$ and its kernel $\mathrm{Ker}^{\mathbb{C}} p_q \subset \mathbb{C}_{T_S}^{2^{[n/2]}}$. We are concerned with $r_{S,M}\mathrm{Ker}^{\mathbb{C}} p_q \subset r_{S,M}\mathbb{C}_{T_S}^{2^{[n/2]}}$, but it is obvious that $r_{S,M}\mathrm{Ker}^{\mathbb{C}} p_q$ can be identified with $\mathbb{C}_S^{2^{[n/2]}}$, because its element is a function only of S. Then we have similar relation of $\mathrm{Ker}^{\mathbb{C}} p_q$ in Proposition 5.2.

After we suppress a normal translation freedom in $\mathcal{H}_{p_q}$, we might choose a position q and make q vanish. Thus we will give our definition of the submanifold Dirac operator.

Definition 5.1. *We define the submanifold Dirac operator over S in M by,*

$$\slashed{D}_{S \hookrightarrow M} := r_{S,M}(\eta_{\mathrm{sa}}(\slashed{D}_M)|_{\mathrm{Ker}(p_q)}),$$

as an endomorphism of Clifford submodule $r_{S,M}\mathrm{Ker}(p_q) \subset r_{S,M}\tilde{\mathrm{Cliff}}_M$*, i.e.,*

$$\slashed{D}_{S \hookrightarrow M} : r_{S,M}\mathrm{Ker}(p_q) \to r_{S,M}\mathrm{Ker}(p_q).$$

Further we extend its domain and region to $r_{S,M}\mathrm{Ker}^{\mathbb{C}} p_q$ *or* $\mathbb{C}_S^{2^{[n/2]}}$*;*

$$\slashed{D}_{S \hookrightarrow M} : \mathbb{C}_S^{2^{[n/2]}} \to \mathbb{C}_S^{2^{[n/2]}}.$$

Here we note that the first restriction $|_{\mathrm{Ker}(p_q)}$ is as an operator but the second one $r_{S,M}$ is associated with a sheaf theory [[15]].

In order to find the extension for $\not{D}_{S \hookrightarrow M}$ over $\mathbb{C}_S^{2^{[n/2]}}$ we need an explicit representation of the Dirac operator. For the case that M is the euclidean space, we find a natural frame to represents the Clifford objects explicitly. However local parameter of M is not privileged in general. Thus we introduce another Clifford ring sheaf isomorphic to $r_{S,M}\mathrm{CLIFF}_M$ and find its explicit isomorphism using an element of Clifford group.

Let us introduce a vector sheaf $\mathbb{R}_S^n$ related to G-action and a sheaf morphism $\iota_{\mathbb{R}^n,S} : \Theta_S \to \mathbb{R}_S^n$ and an isomorphism $\mu_{\mathbb{R}^n,M} : r_{S,M}\Theta_M \to \mathbb{R}_S^n$. Using this, we will investigate the Clifford objects over S and ones over M with $r_{S,M}$ before we deal with the Dirac operator.

Using the vector sheaf $\mathbb{R}_S^n$, we construct a Clifford ring sheaf $\mathrm{CLIFF}(\mathbb{R}_S^n)$ over S generated by a linear sheaf morphism $\gamma_{\mathbb{R}_S^n} : \mathbb{R}_S^n \to \mathrm{CLIFF}^1(\mathbb{R}_S^n)$. Similarly we could define its representation module $\mathrm{Cliff}(\mathbb{R}_S^n)$ and its Clifford groups $\mathrm{CG}(\mathbb{R}_S^n)$.

We have an isomorphism $\mu_{\mathbb{R}^n,M}{}^\flat : r_{S,M}\mathrm{CLIFF}_M \to \mathrm{CLIFF}(\mathbb{R}_S^n)$ and one between the Clifford groups $\mathrm{CG}(\mathbb{R}_S^n)$ and $r_{S,M}\mathrm{CG}_M$. By identifying $\mathrm{CG}(\mathbb{R}_S^n)$ with $r_{S,M}\mathrm{CG}_M$, $\mu_{\mathbb{R}^n,M}{}^\flat$ is realized as $\mu_{\mathbb{R}^n,M}{}^\flat(c) = \tau^{-1}c\tau$ for $c \in \mathrm{CLIFF}_M$ and $\tau \in r_{S,M}\mathrm{CG}_M$. Then we also have its representation $\mathrm{Cliff}(\mathbb{R}_S^n)$, and an isomorphism $\mu_{\mathbb{R}^n,M}{}^\sharp : r_{S,M}\mathrm{Cliff}_M \to \mathrm{Cliff}(\mathbb{R}_S^n)$.

The $\iota_{\mathbb{R}^n,S}$ induces a ring homomorphism $\iota_{\mathbb{R}^n,S}{}^\flat : \mathrm{CLIFF}_S \to \mathrm{CLIFF}(\mathbb{R}_S^n)$ by its generator corresponding to $u^{(k)} \in \Theta_S$ by $\gamma_S(u^{(k)}) \mapsto \gamma(\iota_{\mathbb{R}^n,S}(u^{(k)}))$. Similarly we have $\iota_{M,S}{}^\flat : \mathrm{CLIFF}_S \to r_{S,M}\mathrm{CLIFF}_M$. The $\iota_{\mathbb{R}^n,S}{}^\flat = \mu_{\mathbb{R}^n,M}{}^\flat \iota_{M,S}{}^\flat$ and $\iota_{M,S}{}^\flat$ induce the induced and restrict representations modeling ones in §2 such that

$$\mathrm{Ind}^{\iota_{\mathbb{R}^n,S}}{}_S^{\mathbb{R}^n} : \mathrm{Cliff}_S \to \mathrm{Cliff}(\mathbb{R}_S^n), \quad \mathrm{Res}^{\iota_{\mathbb{R}^n,S}}{}_S^{\mathbb{R}^n} : \mathrm{Cliff}(\mathbb{R}_S^n) \to \mathrm{Cliff}_S,$$

$$\mathrm{Ind}^{\iota_{M,S}}{}_S^M : \mathrm{Cliff}_S \to \mathrm{Cliff}_M, \quad \mathrm{Res}^{\iota_{M,S}}{}_S^M : \mathrm{Cliff}_M \to \mathrm{Cliff}_S,$$

are connected by natural relations,

$$\mathrm{Ind}^{\iota_{M,S}}{}_S^M = \tau^{-1}\mathrm{Ind}^{\iota_{\mathbb{R}^n,S}}{}_S^{\mathbb{R}^n}, \quad \mathrm{Res}^{\iota_{M,S}}{}_S^M = \mathrm{Res}^{\iota_{\mathbb{R}^n,S}}{}_S^{\mathbb{R}^n}\tau.$$

For every $u \in \Gamma(pt, r_{S,M}\Theta_M)$, $v \in \Gamma(pt, \Theta_S)$, $\psi_u := \mathfrak{i}_M(u)$, and $\psi_{S,u} := \mathrm{Res}^{\iota_{M,S}}{}_S^M \psi_u$, as we showed in (2.6), the Frobenius reciprocity shows

$$(5.2) \qquad \begin{aligned} g_M(\iota_{M,S}(v), u) &= \langle \overline{\psi}_u, \ \gamma_M(g_M(\iota_{M,S_*}(v)))\psi_u \rangle_{\mathrm{Cliff}_M} \\ &= \langle \overline{\psi}_{S,u}, \ \gamma_S(g_S(v))\psi_{S,u} \rangle_{\mathrm{Cliff}_S}. \end{aligned}$$

As in (2.7), by letting $\psi_{\tau u} := \tau\psi_u \in \Gamma(pt, r_{S,M}\mathrm{Cliff}_{\mathbb{R}^n_S})$, we have $\psi_{S,u} = \mathrm{Res}^{\iota_{\mathbb{R}^n,S}\mathbb{R}^n_S}\psi_{\tau u}$ and (5.2) becomes

$$
(5.3) \quad
\begin{aligned}
\langle \overline{\psi}_{\tau u},\ \gamma_{\mathbb{R}^n}(g_{\mathbb{R}^n}(\iota_{\mathbb{R}^n S*}(v)))\psi_{\tau u}\rangle_{\mathrm{Cliff}(\mathbb{R}^n_S)} &= \langle \overline{\psi}_{S,u},\ \gamma_S(g_S(v))\psi_{S,u}\rangle_{\mathrm{Cliff}_S} \\
&= g_S(v, \pi_{SM}(u)),
\end{aligned}
$$

where $\pi_{S,M} : r_{S,M}\Theta_M \to \Theta_S$ is given by $g_M(\iota_{M,S}(v), u) = g_S(v, \pi_{SM}(u))$, which is the simplest Frobenius reciprocity; we use its lift to the Clifford modules. These give the data of $\mathrm{Gr}_S^{(n,k)}$ and immersion $\iota_{M,S}$, which are our purpose.

As we find relations among the Clifford objects over S and ones over M with $r_{S,M}$, we step to the consideration of the Dirac operator. In order to obtain the relation (5.3), we will use the Dirac operator $\slashed{D}_{S\hookrightarrow M}$. However we did not give its explicit representation yet. In order to determine an explicit representation of the Dirac operator, using $\tau \in \mathrm{CG}(\mathbb{R}^n_S)$ which connects $\mathrm{CLIFF}(\mathbb{R}^n_S)$ and $r_{S,M}\mathrm{CLIFF}_M$ as mentioned above, we will define the Dirac operator defined over $\mathrm{Cliff}(\mathbb{R}^n_S)$

$$
\slashed{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n,S}} := \tau\ \slashed{D}_{S\hookrightarrow M}\tau^{-1}.
$$

Proposition 5.3. *The submanifold Dirac operator of S in M can be expressed by*

$$
(5.4) \quad \slashed{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n,S}} = \iota_{\mathbb{R}^n,S}^\sharp(\slashed{D}_S) + \frac{1}{2}\gamma^{\dot{\alpha}}\mu_{\mathbb{R}^n,M_*}\Gamma_{\dot{\alpha}}.
$$

where $\slashed{D}_S$ is the proper Dirac over S, $\Gamma_{\dot{\alpha}}/k$ is the mean curvature vector of $b^{(n)\dot{\alpha}}$ [[34] p.119] of S and $\gamma^{\dot{\alpha}} := \gamma_{\mathbb{R}^n_S}(\mu_{\mathbb{R}^n,M_*}(dq^{\dot{\alpha}}))$.

Proof. First we note that $\eta_{\mathrm{sa}}(\slashed{D}_M)$ has a decomposition,

$$
\eta_{\mathrm{sa}}(\slashed{D}_M) = \slashed{D}_M^{\|} + \slashed{D}_M^{\perp},
$$

where $\slashed{D}_M^{\perp} := \gamma_M(dq^{\dot{\alpha}})\partial/\partial q^{\dot{\alpha}}$ and $\slashed{D}_M^{\|}$ does not include the normal derivative $p_{\dot{\alpha}}$. $\slashed{D}_M^{\perp}$ vanishes at $\mathrm{Ker}(p_q)$ and at $\mathrm{Ker}^{\mathbb{C}}(p_q)$. Due to the constructions, $\iota_{M,S}(\gamma_S(e^{(k),\alpha}))$ and $\gamma_M(dq^{\dot{\alpha}})$ become generator of the CLIFF_{T_S} at sufficiently vicinity of S. A direct computation shows that the following relation holds

$$
r_{S,M}\left(\slashed{D}_M^{\|}\right) - \tau^{-1}\iota_{\mathbb{R}^n,S}^\sharp(\slashed{D}_S)\tau = \frac{1}{2}r_{S,M}\left(\gamma_M(dq^\alpha)\Gamma_\alpha\right).
$$

The geometrical independence due to (3.2) and direct computations give above the result. Using $\iota_{\mathbb{R}^n,S}$ and $\mu_{\mathbb{R}^n,M}$, we have the result. $\qquad\square$

Remark 5.2. (1) $\sqrt{-1}\iota_{\mathbb{R}^n,S}^\sharp(\slashed{D}_S)$ is a formal self-adjoint for a L^2-type integral of the Clifford module over S because from the definition, $\sqrt{-1}\slashed{D}_S$ is self-adjoint for the integral over S and $\iota_{\mathbb{R}^n,S}^\sharp$ is $*$-morphism. On the other hand, $\sqrt{-1}\ \slashed{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n,S}}$ is not self-adjoint because of the extra term and the self-adjointness of $\sqrt{-1}\iota_{\mathbb{R}^n,S}^\sharp(\slashed{D}_S)$.

(2) Here we comment on the submanifold Dirac operators defined by C. Bär [[2], Lemma 2.1]. The Dirac operator $\tilde{D}$ in [2] corresponds to our $\sqrt{-1}\iota_{\mathbb{R}^n,S}^{\sharp}(\not{D}_S)$ whereas the Dirac operator $\hat{D}$ in [2] corresponds to our $\sqrt{-1}\ \not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n,S}}$. In [9] the generalized Weierstrass relation is studied using the Dirac operator which is the same as Bär's. Further we note that in [2], $\tilde{D}$ is mainly investigated, whereas we consider $\sqrt{-1}\ \not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n,S}}$, which is not self-adjoint.

(3) Ginoux and Morel [10], and Oussama and Zhang [27] dealt with the same operator $\sqrt{-1}\ \not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n,S}}$ but their studies started from the definition of $\sqrt{-1}\ \not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n,S}}$. They did not mention answer why they employ the definition in detail, at least, from viewpoint of the submanifold quantum mechanics.

(4) It is clear why $\sqrt{-1}\ \not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n,S}}$ has extra non-trivial term. It appears due to the requirement that the projection ϖ_{p_q} should be the self-adjointness, which is the same as the requirement that the isomorphism φ should preserve for the action of ϖ_{p_q}. These are essential to submanifold quantum mechanics [23].

Now we will give our main theorem:

Theorem 5.1. *Fix the data of* Cliff_M *i.e., its base* $\langle c_M^a \rangle_{a=1,\cdots,2^{[n/2]}}$*, and a morphism* $\mathfrak{i}: \Theta_M \to \mathrm{Cliff}_M^{pr}$*. Let a point pt be in S immersed in M. Let* $\mathbb{C}_S^{2^{[n/2]}}$ *be a sheaf of complex vector bundle over S with rank $2^{[n/2]}$. A set of germs of* $\Gamma(pt, \mathbb{C}_S^{2^{[n/2]}})$ *satisfying the submanifold Dirac equation,*

$$\sqrt{-1}\ \not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n,S}}\psi = 0 \quad at \quad pt,$$

is given by $\{b_a\psi^a \mid a = 1, \cdots, 2^{[n/2]},\ b_a \in \mathbb{C}\}$ *such that elements satisfy the orthonormal relation as $\mathbb{C}$-vector space;*

$$\varphi_{pt}(\psi^a)\psi^b = \delta_{a,b} \quad at \quad pt.$$

Then followings hold:

(1) $\langle \psi^a \rangle_{a=1,\cdots,2^{[n/2]}}$ *is a base of* $\Gamma(pt, \mathrm{Cliff}(\mathbb{R}_S^n))$*. There exists an isomorphism* $\mu_{\mathbb{R}^n,M}{}^{\sharp}: r_{S,M}\mathrm{Cliff}_M \to \mathrm{Cliff}(\mathbb{R}_S^n)$ *related to* $\tau \in \Gamma(pt, r_{S,M}\mathrm{CG}_M)$ *satisfying* $\psi^a = \tau c_M^a$ *($a = 1,\cdots,2^{[n/2]}$) by identifying* $\mathrm{Cliff}(\mathbb{R}_S^n)$ *with* $r_{S,M}\mathrm{Cliff}_M$*. τ corresponds to an element of* $\mathrm{SO}(r_{S,M}\Theta_M)$ *as a representative element of* $\mathrm{Gr}_S^{(n,k)}$*.*

(2) *For every* $u \in \Gamma(pt, r_{S,M}\Theta_M)$*, let* $\psi_u := \mathfrak{i}_M(u) \in \Gamma(pt, r_{S,M}\mathrm{Cliff}_M^{pr})$*,* $\psi_{u,S} := \tau\psi_u \in \Gamma(pt, r_{S,M}\mathrm{Cliff}_M^{pr})$ *using τ of (1), and* $\overline{\psi}_{u,S} := \varphi(\psi_u)\tau^{-1} \in \Gamma(pt, \overline{r_{S,M}\mathrm{Cliff}_M^{pr}})$*. Then for every* $v \in \Gamma(pt, r_{S,M}\Theta_S)$*, the following relation holds:*

(5.5) $$\langle \overline{\psi}_{u,S}[\iota_{\mathbb{R}^n,S}{}^{\flat}(\gamma_S(g_S(v)))]\psi_{u,S}\rangle_{\mathrm{Cliff}(\mathbb{R}_S^n)} = g_M(\iota_{M,S}(v), u).$$

This value brings us the local data of immersion $\iota_{M,S}$*.*

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Proof. Since $\not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n},S}$ is the $2^{[n/2]}$ rank first order differential operator and has no singularity over S due to the construction, a germ of its kernel in $\Gamma(pt, \mathbb{C}_S^{2^{[n/2]}})$ is given by $2^{[n/2]}$ dimensional vector space at each point of S. Since $\not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n},S}$ is defined as an endomorphism of $\mathrm{Ker}^{\mathbb{C}}(p_q) \approx \mathbb{C}_S^{2^{[n/2]}}$. The kernel of the Dirac operator, $\mathrm{Ker}^{\mathbb{C}}(\not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n},S})$ of $\mathbb{C}_S^{2^{[n/2]}}$ has an injection into $\mathrm{Cliff}(\mathbb{R}_S^n)$. There exist $\tau \in \mathrm{CG}(\mathbb{R}_S^n)$ such that $\mu_{\mathbb{R}^n,M}^{\flat-1} : r_{S,M}\mathrm{Ker}^{\mathbb{C}}(\not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n},S}) \to \mathrm{Cliff}_M$.

Let $\not{D}_S^{\perp} := \tau^{-1}\gamma^{\dot{\alpha}}\partial_{\dot{\alpha}}\tau$ at S. From the construction, we have

$$\iota_{M,S_*}\not{D}_{S\hookrightarrow M} + \not{D}_S^{\perp} = r_{S,M}(\eta_{\mathrm{sa}}(\not{D}_D)).$$

Hence $\mathrm{Ker}^{\mathbb{C}}(\not{D}_{S\hookrightarrow M}^{\iota_{\mathbb{R}^n},S})$ is a subset of a kernel of $\tau(r_{S,M}(\eta_{\mathrm{sa}}(\not{D}_M)))\tau^{-1} \subset r_{S,M}\mathbb{C}_M^{2^{[n/2]}}$.

Noting Proposition 5.2, $\tilde{\varphi}_{pt}$ is an isomorphism and $\mathcal{H}_{pq}^{pt}$ gives (5.2) and (5.3). Thus we prove them. □

Remark 5.3. (1) The finial result does not depend upon a choice of $\iota_{\mathbb{R}^n,S}$.

(2) This theorem is based upon the Frobenius reciprocity of Clifford ring sheaves on category of differential geometry as shown in (5.3) and (5.2). We have compared $\mathrm{Ind}_S^{\iota_{\mathbb{R}^n,S}\mathbb{R}^n}\mathrm{Cliff}_S$, which is obtained by using the Dirac operator, with Cliff_M as each germ in Theorem 5.1.

(3) We have assumed that M and S are homeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively. However as our arguments are local, the theorem could be extended to spin manifolds S and M under assumptions on the group action if there is no geometrical obstruction.

(4) With Remark 5.1 and 5.2 (4), it is obvious that the submanifold Dirac operator given in submanifold quantum mechanics represents local immersed geometry. Its essential is that the restriction of the Dirac operator preserving φ in Definition 5.1 consists with the Frobenius reciprocity. It is the answer of the question mentioned in Introduction.

(5) If M and S have natural parameterization $(x^i)_{i=1,\cdots,n}$ and $(s^{\alpha})_{\alpha=1,\cdots,k}$ and S is an analytic submanifold such that

$$x^i(s) = \int_S^s dx^i(s)$$

represents an immersion S in M, it can be expressed as

$$x^i(s) = \int_S^s g_{S,\alpha,\beta}\langle \overline{\psi}_{\partial_{x^i},S}[\iota_{M,S}^{\sharp}(\gamma_S(ds^{\alpha}))]\psi_{\partial_{x^i},S}\rangle_{\mathrm{Cliff}(\mathbb{R}_S^n)} ds^{\beta},$$

where $\partial_{x^i} := \partial/\partial x^i$ using above ψ. This is the generalized Weierstrass relation.

(6) When $M \equiv \mathbb{R}^n$ and $k = 2$, the theorem is reduced to the generalized Weierstrass relation [9, 17, 18, 28]. In the case, $\iota_{\mathbb{R}^n,S}$ is properly determined and identify $\mathbb{R}_S^n$ with $\mathbb{R}^n$. These are closely related to

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the two-dimensional integrable system. Especially, when $\displaystyle{\not{D}}_{S \hookrightarrow M}^{\iota_{\mathbb{R}^n}, S}$ is identified with ${\not{D}}_S$ and $\bar{\partial}$, which correspond to minimal surface cases, it becomes original Weierstrass relation [[7], p.260-7].

(7) As mentioned in [22], we can put the Frenet-Serret torsion field into the Dirac operator.

(8) For $k = 1$ case, Theorem is mere the Frenet-Serret relation [19, 20].

(9) As we showed in [19, 20], the Dirac operator also might give the global properties of the immersion of S, i.e., its topological properties, though we mentioned only local properties in this article. Thus we should investigate the global properties using the submanifold Dirac operator as generalization of [19, 20] in future.

(10) When S is a conformal surface, we may consider the relations along the line of arguments of [3, 17, 30, 31, 32, 28]. For example, we could classify the immersions using the Dirac operator. Furthermore when S has holomorphic properties, we also may give similar arguments.

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